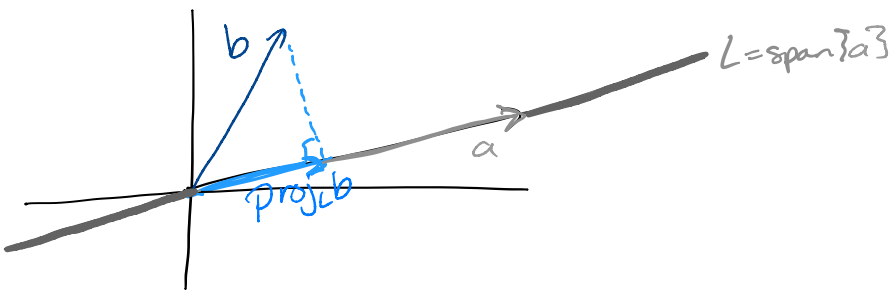
Projections

Let $a \in \mathbb{R}^n$ and $L = \text{Span}\{a\}$.

Let $b \in \mathbb{R}^n$.

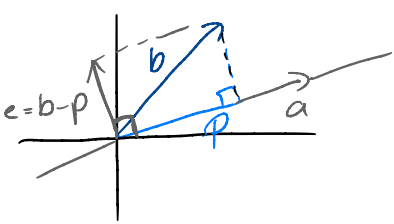
The projection of b onto L is the vector p in L that is closest to b (Notation: $p = \text{proj}_L b$ or $\text{proj}_a b$)



Since $p \in \text{Span}\{a\}$, $p = \hat{x}a$ for some scalar $\hat{x}$. What is $\hat{x}$?

Consider the right triangle

The vector $e = b - p$ ("error") is perpendicular to a.



Hence $a \cdot (b - p) = 0$

$$a \cdot b = a \cdot (\hat{x}a)$$

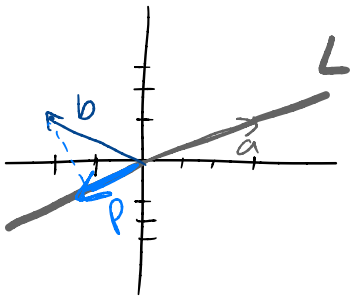
$$a \cdot b = \hat{x}(a \cdot a)$$

$$\Rightarrow \hat{x} = \frac{a \cdot b}{a \cdot a}$$

Thus the projection of b onto $L = \text{span}\{a\}$ is the vector

$$\text{proj}_L b = \frac{a \cdot b}{a \cdot a} a$$

Ex: Project $b = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ onto $L = \text{Span}\left\{\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right\}$ (a)

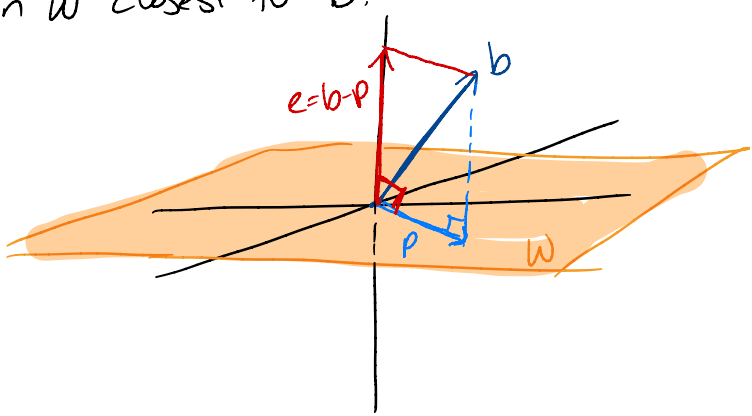


$$\text{proj}_L b = \frac{a \cdot b}{a \cdot a} a = \frac{-5}{10}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -3/2 \\ -1/2 \end{bmatrix}$$

The error vector is $e = b - \text{proj}_L b$

$$= \begin{bmatrix} -1/2 \\ 3/2 \end{bmatrix}$$

Now let W be any subspace of $\mathbb{R}^m$ and $b \in \mathbb{R}^m$, then the projection of b onto W, $p = \text{proj}_W b$, is the vector in W closest to b.



Let $\{a_1, \dots, a_n\}$ be a basis for W.

(recall that in the case W = line, we had $L = \text{Span}\{a\}$)

Since $p \in W$, it is a linear combination of $a_1, \dots, a_n$.

That is, $p = \hat{x}_1 a_1 + \cdots + \hat{x}_n a_n$ for some scalars $\hat{x}_1, \dots, \hat{x}_n$

Let $A = [a_1 \cdots a_n]$.

Then $p = A\hat{x}$, where $\hat{x} = \begin{bmatrix} \hat{x}_1 \\ \hat{x}_n \end{bmatrix}$ (def'n of matrix-vector mult)

As in the case $W = \text{line}$, the error vector $e = b - p = b - A\hat{x}$ is perpendicular to W. Hence $v \cdot (b - A\hat{x}) = 0$ for all $v \in W$.

In particular, $a_i \cdot (b - A\hat{x}) = 0$ for all $1 \le i \le n$.

$$\iff a_i^T (b - A\hat{x}) = 0 \quad \text{(matrix mult)}$$

$$\iff A^T (b - A\hat{x}) = \vec{0}$$

$$\iff A^T b - A^T A\hat{x} = \vec{0}$$

$$\iff A^T A\hat{x} = A^T b.$$

Hence, to find the projection of b onto $W = \text{Span}\{a_1, \dots, a_n\}$

1. Solve $A^T A\hat{x} = A^T b$ where $A = [a_1 \cdots a_n]$
   (this gives the weights of $p = \hat{x}_1 a_1 + \cdots + \hat{x}_n a_n$)
2. Evaluate $\text{proj}_W b = A\hat{x}$

Ex: Let $b = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}$ and $W = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}$

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \implies A^T A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix} \text{ (symmetric!)}$$

$$\text{and } A^T b = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$$

Now solve $\begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix} \hat{x} = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$:

$$\left[\begin{array}{cc|c} 3 & 3 & 6 \\ 3 & 5 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & -3 \end{array}\right] \implies \hat{x} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$$

Hence $\text{proj}_W b = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ -3 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix}$

Intuitively, we expect the projection of b onto W to be unique for all $b \in \mathbb{R}^m$. This means $A^TA\hat{x} = A^Tb$ should have a unique solution for all $b \in \mathbb{R}^m$. This occurs if and only if A^TA is invertible

$\Rightarrow$ $\hat{x} = (A^TA)^{-1}A^Tb$ and $\text{proj}_W b = A(A^TA)^{-1}A^Tb$

Note: if $W = \text{span}\{a\}$, then $A(A^TA)^{-1}A^Tb = \frac{a \cdot b}{a \cdot a} a$

This intuition is correct (i.e. A^TA is invertible)
(proof coming soon)

Ex: Redo previous example

Let $b = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}$ and $W = \text{Span}\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}\right\}$

$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \Rightarrow A^TA = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix}$

$\Rightarrow (A^TA)^{-1} = \frac{1}{6}\begin{bmatrix} 5 & -3 \\ -3 & 3 \end{bmatrix}$

So $\text{proj}_W b = A(A^TA)^{-1}A^Tb$

$= \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 5/6 & -1/2 \\ -1/2 & 1/2 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}$

$= \begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix}$ (same as before)

Connection to Linear Transformations

We can view projection of vectors in $\mathbb{R}^m$ onto W as a linear transformation $T: \mathbb{R}^m \to \mathbb{R}^m$ with image W. The associated matrix is called a projection matrix P.

Since $T(b) = \text{proj}_W b = A\hat{x} = A(A^TA)^{-1}A^Tb \quad \forall b \in \mathbb{R}^m$, the matrix is $\boxed{P = A(A^TA)^{-1}A^T}$

Ex: Let $W = \text{Span}\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}0\\1\\2\end{bmatrix}\right\}$ as above

Then $A = \begin{bmatrix}1&0\\1&1\\1&2\end{bmatrix}$ and $A^TA = \begin{bmatrix}3&3\\3&5\end{bmatrix}$.

Now $(A^TA)^{-1} = \frac{1}{6}\begin{bmatrix}5&-3\\-3&3\end{bmatrix} = \begin{bmatrix}5/6&-1/2\\-1/2&1/2\end{bmatrix}$

So $P = A(A^TA)^{-1}A^T = \begin{bmatrix}1&0\\1&1\\0&2\end{bmatrix}\begin{bmatrix}5/6&-1/2\\-1/2&1/2\end{bmatrix}\begin{bmatrix}1&1&0\\0&1&2\end{bmatrix}$

$$= \frac{1}{6}\begin{bmatrix}5&2&-1\\2&2&2\\-1&2&5\end{bmatrix}$$

Note: Since P projects onto W, which is a plane, $C(P)$ is 2-dim'l. So $\text{rank}\, P = 2$.

More generally, $\text{rank}\, P$ tells us the dimension of the subspace we are projecting onto.

Proof that A^TA is invertible (when columns of A are a basis for W)

Key Fact 1: $N(A^TA) = N(A)$

Proof:

- Let $x \in N(A)$. Then $Ax = \vec{0}$ and so $(A^TA)x = A^T(Ax) = A^T\vec{0} = \vec{0}$. Hence $x \in N(A^TA)$. and so $N(A) \subseteq N(A^TA)$.
- Let $x \in N(A^TA)$. Then $A^TAx = \vec{0}$. Multiplying by x^T gives $x^TA^TAx = x^T\vec{0} = 0$

  ← dot product

  Since $x^TA^TAx = (Ax)^TAx = (Ax) \cdot (Ax) = \|Ax\|^2$

  we have $\|Ax\|^2 = 0 \implies Ax = 0$

  Hence $x \in N(A)$ and so $N(A^TA) \subseteq N(A)$.

Since $N(A) \subseteq N(A^TA)$ and $N(A^TA) \subseteq N(A)$,

we have $N(A) = N(A^TA)$. ∎

Key Fact 2: A^TA is invertible if and only if the columns of A are linearly independent

↑ not necessarily square.

Proof:

A^TA is invertible $\iff N(A^TA) = \{0\}$ (since A^TA is square)

$\iff N(A) = \{0\}$ (by Key Fact 2)

$\iff$ Columns of A are linearly independent ∎

In our setup, since $\{a_1, \dots, a_n\}$ is a basis for W, the matrix $A = [a_1 \cdots a_n]$ has L.I columns. So A^TA is indeed invertible